Four letters written by Julius Plücker to Thomas Archer Hirst[1]
September 1866 –October 1867

September 1866

who made them, to send them to Mr White. The explanations he gives are his not mine.

Let me here repeat a few observations I made at Nottingham.

Each plane intersects a surface of the second degree along a conic; each point is the centre of a circumscribed cone. If the plane is the polar plane of the point, the cone touches the surface along the curve. With regard to a <u>complex</u> of the second degree to each right line corresponds another conjugated line. The surface formed by curves of the complex situated within planes passing through any one of two <u>conjugated</u> lines is at the same time enveloped by cones of the complex the centres of which are placed on the same line. The pole of that line with regard to any of the curves falls within the conjugated line, the polar plane of the same line with regard to any of the curves passes through the conjugated one. There are two conjugated surfaces belonging to every two conjugated lines. <u>Equatorial</u> surfaces belong to an infinitely distant line, its conjugated line passes through the centre of all curves, generating the surface. In a given complex there are three conjugated equatorial surfaces corresponding to any three conjugated diameters of a certain surface of the second degree.

I shall occasionally have models made of *three* such surfaces. The surfaces I send, all equatorial ones, have no relation one to another.

Bonn, *17th October 1866*

My dear Sir,

I can easily comply with your request concerning the models you have received. All belong to that class of surfaces I have called equatorials, formed by curves of the second class confined in parallel planes, and enveloped by cylinders of the second order having their axes parallel to the planes. The surfaces, both of the 4th class and the 4th order, depend generally upon 13 constants.

In admitting ordinary axes of coordinates *OX*, *OY*, *OZ* you may determine by x any plane parallel to *Y Z*, and within this plane by means of line coordinates u, v, w a curve by the equation

$$Aw^2 + (B \quad 2Rx + Fx^2)v^2 \quad 2(G + Ox)uv + (C \quad 2Ux + Ex^2)u^2 = 0$$

which admits one arbitrary constant corresponding with the arbitrary position on *OX* may be said, by regarding x as variable, to represent the equatorial surface: *OX* being its diameter. By introducing, instead of u, v, w the coordinates y, z you will obtain in x, y, z the equation of the same surface. The same equation may be represented by the equation

$$(E\mathrm{tang}^2\delta \quad 2C\mathrm{tang} \quad + F)x^2 + Dz^2 + 2(U\mathrm{tang}^2 \quad O\mathrm{tang} \quad + R)x + C(\mathrm{tang}^2 \quad + B) = 0$$

[Here I must signalise a mistake made in connecting the single parts of the two last models. The singular lines are not double lines. Each singular line is an infinite one, consisting of two parts joining in one point. Accordingly for instance, the extreme parts of the last model are to be turned through an angle of 180 .]

I am, my dear Sir, highly indebted to you by the kind offer to have my researches printed by the R.S. I could transmit to you an Enumeration of about 800 different species of surfaces, both equatorial and meridional, of the fourth based on the same principles made use of in my Theory of algebraical curves. But that would disguise the true object of my inquiries, the nodal point of which is the conception of a complex, corresponding to four variables, while a surface corresponds to three and a plane curve to two variables only.

I think the following instance will explain my meaning. To any given line *A* corresponds an equatorial surface *I* generated by curves confined in planes (*a*) conjugated to *A* and enveloped by cylinders the axes of which are parallel to the planes (*a*), as well as a meridional surface *II* enveloped by cones the centre of which fall within *A* and generated by curves confined in planes passing through *A*. Having before you the equations of both surfaces *I* & *II*, you meet in these equations all constants of the complex, in a linear way except one. Hence it follows that the complex is determined in a linear way by two such combined surfaces and one of its lines *B*. Indeed, trace through *B* any plane (*b*) intersecting *A* in any point *α*. The point is the centre of an enveloping cone, two generatrices of which are confined within the plane (*b*). Besides the same plane (*b*) meets an enveloping cylinder, the axis of which is parallel to the intersection of (*b*) with the conjugated planes (*a*), along two generatrices. Thus are obtained within the plane (*b*) five lines belonging to the complex, therefore the curve itself, the tangents of which are lines of the complex.

Again let (*B*) be any point of *B*. There is one conjugated plane (*a*), and another passing through the given line *A*, which meet in the point (*B*). Trace through this point two tangents to each curve confined in the two planes. Both couples of tangents, if joined to *B*, determine a cone, generated by lines of the complex, the centre of which is *B*.

You may give a great extent to construction of this kind, and, if you like, proceed in a pure geometrical way: starting from the general definition of a complex of the second degree.

Having now, after long preparatory attempt, a full geometrical oversight of complexes of the second degree and having met with an analytical method, surpassing by far, with regard to simplicity and symmetry whatever I before expected. I intend to publish a first volume next year, containing a complete classification and discussion of complexes, which involve the discussion of equatorial and meridional surfaces. The number of constants of a complex is 20nts

There is another Chapter treated by me with full success: The theory of linear polar complexes with regard to complexes of the second degree, as well as of tangent complexes.
Other Chapters has been reserved.

I shall be most happy to give any information whatever you request, but after all I think it desirable not to publish details now. I think a part of the developments which I have before me, if translated into French, would satisfy a question put by the Institute. I hesitated for a moment, by my indolence prevailed, the more as Mr. Chastles would recognise the author.

If I were twenty years younger I could hope to see the principles laid down in the two printed Papers, fully developed. Now, one of my former pupils will try to work out the last *x* of my last Paper into a treatise of Mechanics. I shall be glad if he succeeds and in this case assist him. A short elementary account of the analytical methods employed by myself in former Papers would be necessary or desirable as an introductory work. It would not be wise to overcrowd myself with heterogeneous work at present. When I have finished a geometrical volume I intend to concentrate my attention to the contents of the two first *xx* of my last Paper and its application to molecular Physics. Then I may at first not undertake to write a systematical work and I shall be happy to present elaborate Papers on especial points to the R.S. I wished to satisfy as much as possible your request, but do not know if I succeeded.

Believe me
My dear Sir
truly yours
Plücker

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Bonn, *11 November 1866*

My dear Sir!

Many thanks for your friendly communication of the decision in my favour of the Council of the Royal Society! I feel myself highly indebted to you for your personal interest and assistance. When my researches are published, I hope the Council will be justified in the honour it has now conferred.

I received the intelligence, first kindly noticed to me by you, soon after in a kind letter from the President.

Hoping to hear from you again.

I am, my dear Sir, most truly yours
Plücker

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